

HEAT KERNELS, SMOOTHNESS ESTIMATES AND EXPONENTIAL DECAY

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ABSTRACT. In this article, we establish Gaussian decay for the \square_b -heat kernel on polynomial models in \mathbb{C}^2 . Our technique attains the exponential decay via a partial Fourier transform. On the transform side, the problem becomes finding quantitative smoothness estimates on a heat kernel associated to the weighted $\bar{\partial}$ -operator on $L^2(\mathbb{C})$. The bounds are established with Duhamel's formula and careful estimation.

1. INTRODUCTION

The purpose of this article is to prove Gaussian decay for the \square_b heat kernel on polynomial models in \mathbb{C}^2 and introduce a class of estimates called quantitative smoothness estimates. We develop a new method for obtaining exponential decay via the Fourier transform as our newly developed quantitative smoothness estimates characterize such functions. We are then able to show that the kernel associated to a weighted $\bar{\partial}$ -operator on \mathbb{C} satisfies a number of quantitative smoothness estimates, and this allows us to recover the Gaussian decay estimate for the \square_b heat kernel.

1.1. Polynomial models in \mathbb{C}^2 .

Definition 1.1. A **polynomial model** $M \subset \mathbb{C}^2$ is a manifold of the form

$$M = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = p(z)\}$$

where $p : \mathbb{C} \rightarrow \mathbb{R}$ is a subharmonic, nonharmonic polynomial.

M is the boundary of an unbounded pseudoconvex domain called a polynomial model domain. For example, if $p(z) = |z|^2$, then M is the Heisenberg group \mathbb{H}^1 and is the boundary of the Siegel upper-half space. $M \cong \mathbb{C} \times \mathbb{R}$ with the identification $(z, t + ip(z)) \longleftrightarrow (z, t)$. We will not distinguish $M \subset \mathbb{C}^2$ with its image $\mathbb{C} \times \mathbb{R}$. The tangential Cauchy-Riemann operator $\bar{\partial}_b$ on M can be identified with the vector field

$$\bar{L} = \frac{\partial}{\partial \bar{z}} - i \frac{\partial p}{\partial \bar{z}} \frac{\partial}{\partial t}$$

and $\bar{\partial}_b^*$ on M can be identified with the vector field

$$L = \frac{\partial}{\partial z} + i \frac{\partial p}{\partial z} \frac{\partial}{\partial t}.$$

The Kohn Laplacian $\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ is then identified with $\square_b = -\bar{L}L$ on $(0, 1)$ -forms and $\square_b = -L\bar{L}$ on functions.

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The $\bar{\partial}_b$ -complex on unbounded CR-manifolds is a relatively unexplored subject, and polynomial models provide a model case to study. In addition, polynomial models provide a good local approximation to a CR manifold of finite type and have been used to prove local results in dimension 3, see e.g., [Chr89]. An advantage of working with polynomial models is that the nonisotropic control metric is globally defined [NSW85]. This is one reason that, with notable exceptions such as Kang's work closed range of $\bar{\partial}_b$ on weighted L^2 when $p(z)$ is radial [Kan89], a major focus of the analysis is establishing pointwise estimates on integral kernels (in terms of the control metric) [NRSW89, NS01a, NS06, Rai06b]. As mentioned above, the prototype polynomial model is the Heisenberg group. Analysis on it, however, is aided by the fact that it is a Lie group, whereas the generic polynomial model lacks any group structure.

1.2. \square_b -heat kernel. Our goal is to prove pointwise estimates on the \square_b -heat kernel and its derivatives. For $\alpha = (z, t_1), q = (w, t_2) \in \mathbb{C} \times \mathbb{R}$, The \square_b -heat equation is

$$(1) \quad \begin{cases} \frac{\partial u}{\partial s} + \square_b u = 0 & \text{in } (0, \infty) \times \mathbb{C} \times \mathbb{R} \\ u(0, \alpha) = f(\alpha) & \text{on } \{s = 0\} \times \mathbb{C} \times \mathbb{R} \end{cases}$$

As in [NS01a], we solve (1) using the heat semigroup $e^{-s\square_b}$. In particular, there exists the heat kernel $\mathcal{H}_{\tau p}(s, \alpha, \beta)$ that is C^∞ off of the diagonal $\{s = 0, \alpha = \beta\}$ and if $u(s, \alpha) = (e^{-s\square_b} f)(\alpha)$, then

$$u(s, \alpha) = \int_{\mathbb{C} \times \mathbb{R}} \mathcal{H}_{\tau p}(s, \alpha, \beta) f(\beta) d\beta$$

and u solves (1).

Solving the \square_b -heat equation has many applications to the theory of \square_b . In particular, the spectral theorem for unbounded operators allows us to recover the Szegő kernel S as $S = \lim_{s \rightarrow \infty} e^{-s\square_b}$ and the relative fundamental solution which is given by $\int_0^\infty e^{-s\square_b} (I - S) ds$. Moreover, one method to bound the number of eigenvalues below a fixed threshold requires estimates on the trace (in the operator sense) of the heat kernel for small time.

In [NS01a], Nagel and Stein prove that the heat kernel $\mathcal{H}_{\tau p}(s, \alpha, \beta)$ satisfies rapid decay, and our goal is to present a calculation to improve the decay to exponential decay. Similar results have been obtained in an unpublished result by Nagel and Müller and independently by Street [Str09]. Nagel and Müller adapt the technique of [JSC86] while Street adapts the technique of [Sik04, Rai07]. The disadvantage of the techniques of Nagel/Müller and Street is that they do not seem to generalize to higher dimensions. Our ideas ought to generalize, and we plan to pursue this in a future work.

1.3. Weighted operators on $L^2(\mathbb{C})$. Since the operator \bar{L} is translation invariant in t , we can study \square_b by taking a partial Fourier transform in t . Studying $\bar{\partial}_b$ on polynomial models via the partial Fourier transforms has been a fruitful method [Nag86, Chr91, Has94, Has95, Has98, Rai06a, Rai06b, Rai07, BR09, Rai, BR]

If $f(z, t)$ is a function on $\mathbb{C} \times \mathbb{R}$, we define the partial Fourier transform of f by

$$\hat{f}(z, \tau) = \int_{\mathbb{R}} e^{-it\tau} f(z, t) dt.$$

Under the partial Fourier transform

$$\bar{L} \mapsto \bar{Z}_{\tau p} = \frac{\partial}{\partial \bar{z}} + \tau \frac{\partial p}{\partial \bar{z}} = e^{-\tau p} \frac{\partial}{\partial \bar{z}} e^{\tau p} \quad \text{and} \quad L \mapsto Z_{\tau p} = \frac{\partial}{\partial z} - \tau \frac{\partial p}{\partial z} = e^{\tau p} \frac{\partial}{\partial z} e^{-\tau p}.$$

Similarly, the Kohn Laplacian \square_b on $(0, 1)$ -forms maps to $\square_{\tau p} = -\bar{Z}_{\tau p} Z_{\tau p}$ and \square_b on functions maps to $\tilde{\square}_{\tau p} = -Z_{\tau p} \bar{Z}_{\tau p}$. We will see below that understanding the τ -derivative of the \square_b -heat kernel τ is essential for proving its exponential decay estimates.

Applying the partial Fourier transform to (1), we have the heat equations

$$(2) \quad \begin{cases} \frac{\partial u}{\partial s} + \square_{\tau p, z} u = 0 & \text{on } (0, \infty) \times \mathbb{C} \\ u(0, z) = f(z) & \text{in } \{s = 0\} \times \mathbb{C} \end{cases}$$

and

$$(3) \quad \begin{cases} \frac{\partial \tilde{u}}{\partial s} + \tilde{\square}_{\tau p, z} \tilde{u} = 0 & \text{on } (0, \infty) \times \mathbb{C} \\ \tilde{u}(0, z) = \tilde{f}(z) & \text{in } \{s = 0\} \times \mathbb{C}. \end{cases}$$

Note that u and \tilde{u} are no longer functions of t as they were in (1), and generically, u and \tilde{u} are not functions of τ as we think of τ as a parameter. Let $H_{\tau p}(s, z, w)$ and $\tilde{H}_{\tau p}(s, z, w)$ be the heat kernels associated to (2) and (3), respectively. It turns out that these heat equations are dual to one another in the following sense: if

$$\tilde{\square}_{\tau p, z} = -Z_{\tau p, z} \bar{Z}_{\tau p, z},$$

then

$$\square_{-\tau p, z} = \overline{\tilde{\square}_{\tau p, z}}.$$

This equality, coupled with the fact that $\tilde{\square}_{\tau p, z}$ is self-adjoint, forces

$$(4) \quad H_{-\tau p}(s, z, w) = \overline{\tilde{H}_{\tau p}(s, z, w)} = \tilde{H}_{\tau p}(s, w, z).$$

In other words, the roles and $\bar{Z}_{\tau p}$ and $Z_{\tau p}$ switch when $\tau < 0$. This is a key equality for handling both the $\tau < 0$ case as well as the case when $\square_b = -L\bar{L}$. See Remark 2.2 for details.

1.4. Outline of the article. In Section 2, we introduce notation and formulate the Gaussian decay result on polynomial models, Theorem 2.1. Generically, the exponential decay of the \square_b -heat kernel is of the form $e^{-a|t|^{1/\beta}}$ where $\beta \geq 1$. Since we are using the Fourier transform to recover the estimates, we need to find a condition that is tractable across the transform. To do this, we characterize $e^{-a|t|^{1/\beta}}$ in terms of $\| |t|^\ell e^{-a|t|^{1/\beta}} \|_{L^\infty(\mathbb{R})}$ for $\ell \geq 0$ in the spirit of [GS67]. This leads to estimates on the Fourier transform side that we call quantitative smoothness estimates. This is the content of Section 3. In Section 4, we recast the Gaussian decay in terms of the quantitative smoothness estimates. In Section 5, we formulate the main result on the quantitative smoothness estimate of the $\square_{\tau p}$ -heat kernel, Theorem 5.1, and show that this result implies Theorem 2.1. To establish the estimates of Theorem 5.1, we combine Duhamel's principle and a recursion to find a formula for the τ -derivatives of the $\square_{\tau p}$ -heat kernel. This is the content of Section 6. In Section 7, we prove Theorem 5.1.

2. RESULTS

2.1. The control metric on M . In [NSW85], Nagel et. al. prove the existence of the control metric on manifolds such as M . We need to introduce some quantities to write down an equivalent size to the metric (see [NSW85, NS01a, NS01b] for details). Let

$$T(w, z) = -2 \operatorname{Im} \left(\sum_{j \geq 1} \frac{1}{j!} \frac{\partial^j p(z)}{\partial z^j} (w - z)^j \right).$$

and

$$a_{jk}^z = \frac{1}{j!k!} \frac{\partial^{j+k} p(z)}{\partial z^j \partial \bar{z}^k}, \quad \Lambda(z, \delta) = \sum_{j,k \geq 1} |a_{jk}^z| \delta^{j+k}, \quad \mu(z, \delta) = \inf_{j,k \geq 1} \left| \frac{\delta}{a_{jk}^z} \right|^{\frac{1}{j+k}}.$$

The functions μ and Λ are relative inverses in the sense that

$$\mu(z, \Lambda(z, \delta)) \sim \Lambda(\mu(z, \delta)) \sim \delta.$$

We say that $A \sim B$ if there exists a global constant c so that $\frac{1}{c}A \leq B \leq cA$. For points $\alpha = (z, t_1)$ and $\beta = (w, t_2)$, the control metric on M is equivalent to (with an abuse of notation)

$$d(\alpha, \beta) = d(z, w, t_1 - t_2) = |z - w| + \mu(z, t_1 - t_2 + T(z, w)),$$

and with this distance, the volume of a ball of radius δ , $B_d(\alpha, \delta)$ is

$$|B_d(\alpha, \delta)| \sim \delta^2 \Lambda(z, \delta).$$

Since $|B_d(\alpha, \delta)|$ does not depend on t , we sometimes engage in a small abuse of notation and write $|B_d(z, \delta)|$ in lieu of $|B_d(\alpha, \delta)|$. Given points $\alpha, \beta \in \mathbb{C} \times \mathbb{R}$ as above, the volume of the ball centered at p of radius $d(\alpha, \beta)$ is denoted $V(\alpha, \beta) = V(z, w, t_1 - t_2)$ and

$$\begin{aligned} V(z, w, t_1 - t_2) &= d(z, w, t_1 - t_2)^2 \Lambda(z, d(z, w, t_1 - t_2)) \\ &\sim \max \left\{ |z - w|^2 \Lambda(z, |w - z|), \mu(z, t_1 - t_2 + T(w, z))^2 |t_1 - t_2 + T(w, z)| \right\}. \end{aligned}$$

As a consequence of the “twist”, $T(w, z)$, the derivative in τ is the twisted derivative

$$M_{\tau p}^{z,w} = e^{i\tau T(w,z)} \frac{\partial}{\partial \tau} e^{-i\tau T(w,z)} = \frac{\partial}{\partial \tau} - iT(w, z),$$

as $-i(t + T(w, z))\varphi \mapsto M_{\tau p}^{z,w} \hat{\varphi}(\tau)$.

2.2. Results. For the remainder of the paper, consider z and w as fixed points in \mathbb{C} . Let $J = (j_0, \dots, j_k) \in \{0, 1\}^k$ be a multiindex. We set $X^J = X^{j_0} \dots X^{j_k}$ where $X^0 = L$ and $X^1 = \bar{L}$. We now present our main theorem on polynomial models.

Theorem 2.1. *Let $\mathcal{H}_{\tau p}(s, \alpha, \beta)$ be the \square_b -heat kernel associated to (1). Let J and J' be multiindices. There exists positive constants $C, c > 0$ so that*

$$(5) \quad |X_\alpha^J X_\beta^{J'} \mathcal{H}_{\tau p}(s, \alpha, \beta)| \leq C \frac{e^{-c \frac{d(\alpha, \beta)^2}{s}}}{d(\alpha, \beta)^{|J|+|J'|} V(\alpha, \beta)}$$

for all α and $\beta \in M$ and $s > 0$. If $X_\alpha^J X_\beta^{J'} \frac{\partial^j}{\partial s^j} S(\alpha, \beta) = 0$ where $S(\alpha, \beta)$ is the Szegő kernel, then

$$(6) \quad \left| X_\alpha^J X_\beta^{J'} \frac{\partial^j}{\partial s^j} \mathcal{H}_{\tau p}(s, \alpha, \beta) \right| \leq C \frac{e^{-c \frac{d(\alpha, \beta)^2}{s}}}{s^{j+\frac{1}{2}(|J|+|J'|)} |B_d(\alpha, \sqrt{s})|}$$

for all α and $\beta \in M$ and $s > 0$.

Remark 2.2. There are several reductions that we can make.

- (i) The bounds for $|X_\alpha^J X_\beta^{J'} \mathcal{H}_{\tau p}(s, \alpha, \beta)|$ when $d(\alpha, \beta) \sim |z - w|$ are proved in [Rai].
- (ii) Notice that if $s \geq d(\alpha, \beta)^2$, then $\exp(-c_0 d(\alpha, \beta)^2/s) \sim 1$ and provides no decay as $s \rightarrow \infty$. Consequently, the bounds when $s \geq d(\alpha, \beta)^2$ are a consequence of [NS01a] or [Rai].
- (iii) The estimate (6) is only better than (5) for large s , i.e., when $s \geq d(\alpha, \beta)^2$. In this case, there is decay in s that is simply not present if $X_\alpha^J X_\beta^{J'} \frac{\partial^j}{\partial s^j} S(\alpha, \beta) \neq 0$ as $\lim_{s \rightarrow \infty} e^{-s \square_b} = S$. When $X_\alpha^J X_\beta^{J'} \frac{\partial^j}{\partial s^j} S(\alpha, \beta) = 0$, the decay in (6) occurs because the derivative of the kernel of heat semigroup $e^{-s \square_b}$ will coincide with the derivative of the kernel of the semigroup $e^{-s \square_b}(I - S)$. The estimates in (6) follow immediately from (5) and the estimates for the kernel of $e^{-s \square_b}(I - S)$ proven in [NS01a] (and they can also be obtained from [Rai]). Since the constant c is not sharp, the small time estimate in (6) is equivalent to the small time estimate in (5) (with a slight decrease in c).
- (iv) Theorem 2.1 makes no distinction between $\square_b = -L\bar{L}$ on functions and $\square_b = -\bar{L}L$ on $(0, 1)$ -forms. $-L\bar{L} \mapsto \tilde{\square}_{\tau p}$ while $-\bar{L}L \mapsto \square_{\tau p}$. Because of (4), understanding the $\tau > 0$ cases for $H_{\tau p}(s, z, w)$ and $\tilde{H}_{\tau p}(s, z, w)$ is equivalent to understanding the $\tau < 0$ cases for $\tilde{H}_{\tau p}(s, z, w)$ and $H_{\tau p}(s, z, w)$, respectively.
- (v) Consequently, it suffices to prove Theorem 2.1 when $\square_b = -\bar{L}L$, $s \leq d(\alpha, \beta)^2$, and $d(\alpha, \beta) = \mu(z, t_1 - t_2 + T(w, z))$.

Thus, the content of Theorem 2.1 is to achieve (5) for $d(\alpha, \beta) \sim \mu(z, t_1 - t_2 + T(w, z))$ and $s \leq d(\alpha, \beta)^2$.

3. QUANTITATIVE SMOOTHNESS ESTIMATES

The proof of Theorem 2.1 uses the heuristic that decay on the function side corresponds to smoothness on the transform side. In particular, we need to understand the Fourier transforms of functions that decay like $e^{-a|t|^{1/\beta}}$ when $\beta \geq 1$. To do this, we introduce quantitative smoothness estimates.

Definition 3.1. A smooth function $g : \mathbb{R} \rightarrow \mathbb{C}$ satisfies an L^q -**quantitative smoothness estimate of order β** , abbreviated L^q -QSE, if there exist constants $A, C > 0$ so that for all integers $\ell \geq 0$,

$$\|g^{(\ell)}\|_{L^q(\mathbb{R})} \leq C A^\ell \ell^{\ell\beta}.$$

Here, $g^{(\ell)}$ stands for the derivative of order ℓ of g . If $\beta \leq 1$ and g satisfies an L^∞ -QSE, then g will be in some quasianalytic class and extend holomorphically to a strip ($\beta = 1$) or to an entire function ($\beta < 1$). For $\beta > 1$, the case of interest here, such functions do not lie in any quasianalytic class. This is an immediate consequence of the Denjoy-Carleman Theorem (see [Rud87], Theorem 19.11).

3.1. Explanation of QSE. The ideas for these calculations can be found in [GS67]. From first year calculus, we know that

$$(7) \quad |t|^\gamma e^{-a|t|^{1/\beta}} \leq \left(\frac{\gamma\beta}{ae}\right)^{\gamma\beta}.$$

The surprising fact is that this inequality, if it is true for all γ , is actually equivalent to exponential decay. We have the following proposition from [GS67].

Proposition 3.2. *Let $a, \beta > 0$. Then*

$$(8) \quad e^{-a|t|^{1/\beta}} = \inf_{\gamma \geq 0} \left(\frac{\gamma\beta}{ae|t|^{1/\beta}}\right)^{\gamma\beta} \leq \inf_{\substack{n \geq 1 \\ n \in \mathbb{Z}}} \left\{ 1, \left(\frac{n\beta}{ae|t|^{1/\beta}}\right)^{n\beta} \right\} \leq e^{e\beta/2} e^{-a|t|^{1/\beta}}$$

Proof. We may assume that $t > 0$. Let $\nu_\beta(\xi) = \inf_{\gamma \geq 0} \frac{\gamma\beta}{\xi^\gamma}$ and $A = (\frac{\beta}{ae})^\beta$. Note that $\nu_\beta(t/A)$ is the second term in (8). We have already seen that $e^{-at^{1/\beta}} \leq \nu_\beta(t/A)$. Fix $\xi > 0$. Let $f(\gamma) = \frac{\gamma\beta}{\xi^\gamma}$. Then

$$\log f(\gamma) = \gamma\beta \log \gamma - \gamma \log \xi,$$

so

$$(\log f(\gamma))' = \frac{f'(\gamma)}{f(\gamma)} = \beta + \beta \log \gamma - \log \xi$$

and $(\log f(\gamma))'' = \beta/\gamma > 0$. Thus, the zero of $(\log f(\gamma))'$ corresponds to a minimum of $f(\gamma)$. Also, $(\log f(\gamma_0))' = 0$ means that $\gamma_0 = \frac{1}{e}\xi^{1/\beta}$, so

$$\min_{\gamma \geq 0} \log f(\gamma) = -\frac{1}{e}\beta\xi^{1/\beta}$$

and

$$\min_{\gamma \geq 0} f(\gamma) = e^{-\frac{\beta}{e}|\xi|^{1/\beta}}.$$

Consequently, we see that $\nu_\beta(t/A) = e^{-at^{1/\beta}}$ which establishes the first equality in (8). The first inequality in (8) is obvious, so it remains to show the second inequality.

Let $n_0 = \lceil \gamma_0 \rceil$, i.e., the next largest integer greater than γ_0 . By Taylor's Theorem, there exists γ_1 so that $\gamma_0 \leq \gamma_1 \leq n_0$ so that

$$\log f(n_0) = \log f(\gamma_0) + \frac{\beta}{2\gamma_1}(n_0 - \gamma_0)^2 \leq \log f(\gamma_0) + \frac{\beta}{2\gamma_0}.$$

Thus,

$$f(n_0) \leq e^{-\frac{\beta}{e}|\xi|^{1/\beta}} e^{\frac{\beta e}{2}|\xi|^{-1/\beta}}.$$

If $|\xi| = |t|/A \geq 1$, then $e^{\frac{\beta e}{2}|\xi|^{-1/\beta}} \leq e^{\frac{\beta e}{2}}$ and this establishes the second inequality in (8) in this case. On the other hand, if $|\xi| = |t|/A \leq 1$, then note that

$$\left(\frac{n\beta}{ae|t|^{1/\beta}}\right)^{\beta n} = \frac{n^{n\beta}}{(t/A)^n} > 1 \quad \text{for } n \geq 1.$$

So the left side of the second inequality in (8) is 1. On the other hand, if $t \leq A$, then it is easy to show that the right side of this inequality is greater than 1. This concludes the proof of the second inequality and the proof of the proposition is complete. □

Corollary 3.3. *Let $\beta, A, C > 0$ and $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ be a function that satisfies*

$$\|t^n \varphi\|_{L^\infty(\mathbb{R})} \leq C \left(\frac{n\beta}{ae} \right)^{n\beta}$$

for all integers $n \geq 0$. Then

$$|\varphi(t)| \leq C e^{-a|t|^{1/\beta}}.$$

Corollary 3.3 allows us to connect functions with exponential decay and functions that satisfy quantitative smoothness estimates. In particular, we have our main result for quantitative smoothness estimates.

Theorem 3.4. *Let $\varphi : \mathbb{R} \rightarrow \mathbb{C}$.*

- (1) *Suppose there exist constants $a, \beta > 0$ so that $|\varphi(t)| \leq C e^{-a|t|^{1/\beta}}$. If $A = (\frac{\beta}{ae})^\beta$, then it follows that*

$$|t^n \varphi(t)| \leq C A^n (n\beta)^{n\beta}$$

for all integers $n \geq 0$ and $\hat{\varphi}$ satisfies L^∞ -QSE of order β .

- (2) *Suppose that $\hat{\varphi}$ satisfies L^1 -QSE of order β . Then there exist $A, C > 0$ so that*

$$|t^n \varphi(t)| \leq C A^n (n\beta)^{n\beta}$$

for all integers $n \geq 0$, i.e.,

$$|\varphi(t)| \leq C e^{-a|t|^{1/\beta}}$$

where $A = (\frac{\beta}{ae})^\beta$.

Proof. Proof of (1). Recall that $\|\hat{\varphi}^{(n)}\|_{L^\infty(\mathbb{R})} \leq \|t^n \varphi\|_{L^1(\mathbb{R})}$. It follows that

$$\|\hat{\varphi}^{(n)}\|_{L^\infty(\mathbb{R})} \leq \int_{|t| \leq 1} |t^n \varphi(t)| dt + \int_{|t| > 1} \frac{1}{t^2} |t^{n+2} \varphi(t)| dt = 2C A^n n^{n\beta} (1 + A^2 n^{2\beta}).$$

Next, if $A' = (1 + \epsilon)A > A$, then for large enough n , $(A')^n \geq 1 + A^2 n^{2\beta}$. Consequently, if $C' > C$ is sufficiently large,

$$\|\hat{\varphi}^{(n)}\|_{L^\infty(\mathbb{R})} \leq 2C A^n n^{n\beta} (1 + A^2 n^{2\beta}) \leq C' (A')^n n^{n\beta}.$$

The proof of (2) is immediate from the equality $|t^n \varphi(t)| = \left| \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} \hat{\varphi}^{(n)}(\tau) d\tau \right|$. \square

4. HEAT KERNEL DECAY ESTIMATES IN TERMS OF QSE

Fix $z, w \in \mathbb{C}$. We are interested in the case for which

$$d(z, w, t) \sim \mu(z, t + T(w, z)).$$

Since \square_b is translation invariant in t , if $\alpha = (z, t_1)$, $\beta = (w, t_2)$ and $t = t_1 - t_2$, we can write $\mathcal{H}_{\tau p}(s, \alpha, \beta) = \mathcal{H}_{\tau p}(s, z, w, t)$.

We first prove the estimate (6), with $J = J' = 0$, for $\frac{\partial \mathcal{H}_{\tau p}}{\partial s}(s, z, w, t)$ and then recover the estimate for $\mathcal{H}_{\tau p}(s, \alpha, \beta)$ from it. We wish to find a sufficient condition so that

$$\begin{aligned}
\left| \frac{\partial \mathcal{H}_{\tau p}}{\partial s}(s, z, w, t) \right| &\leq \frac{C}{s B_d(z, \sqrt{s})} e^{-c \frac{\mu(z, t + T(w, z))^2}{s}} \\
&= \sup_{j, k \geq 1} \frac{C}{B_d(z, \sqrt{s})} \exp \left(- \frac{c}{s |a_{jk}^z|^{\frac{2}{j+k}}} |t + T(w, z)|^{\frac{2}{j+k}} \right) \\
(9) \quad &\sim \sum_{j, k \geq 1} \frac{C}{B_d(z, \sqrt{s})} \exp \left(- \frac{c}{s |a_{jk}^z|^{\frac{2}{j+k}}} |t + T(w, z)|^{\frac{2}{j+k}} \right).
\end{aligned}$$

Since $\mathcal{H}_{\tau p}(s, \alpha, \beta) = \overline{\mathcal{H}_{\tau p}(s, \beta, \alpha)}$, we can interchange the roles of z and w in (9) and we will find an estimate that implies (9). Let $\varphi(t) = \frac{\partial \mathcal{H}_{\tau p}}{\partial s}(s, z, w, t)$. By Corollary 3.3, the exponential decay estimate (9) is equivalent to the estimate

$$|(t + T(w, z))^n \varphi(t)| \leq \frac{C A^n}{s B_d(w, \sqrt{s})} \sum_{j, k \geq 1} |a_{jk}^w|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}}$$

for all $n \geq 0$. We can incorporate the $s B_d(w, \sqrt{s})$ into the sum by proving the following:

$$\begin{aligned}
(10) \quad \frac{1}{s^2} \sum_{j, k \geq 1} |a_{jk}^w|^{n-1} s^{(n-1) \frac{j+k}{2}} n^{n \frac{j+k}{2}} &\gtrsim \frac{1}{s B_d(w, \sqrt{s})} \sum_{j, k \geq 1} |a_{jk}^w|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} \\
&\gtrsim \frac{1}{s^2 n^{\frac{1}{2} \deg p}} \sum_{j, k \geq 1} |a_{jk}^w|^{n-1} s^{(n-1) \frac{j+k}{2}} n^{n \frac{j+k}{2}}
\end{aligned}$$

where proportionality constants appearing in \gtrsim only depend on the the number of terms in the sum which is essentially the degree of the polynomial p . Also, since we are allowed geometric terms (i.e., A^n) and $n^{\deg p}$ grows sub-geometrically, (10) allows us to absorb $B_d(w, \sqrt{s})$ into the sum. To prove the inequalities, fix s and observe that

$$(11) \quad s B_d(w, \sqrt{s}) \sim s^2 \sum_{j, k \geq 1} |a_{jk}^w| s^{\frac{j+k}{2}} \sim s^2 \max_{j, k \geq 1} |a_{jk}^w| s^{\frac{j+k}{2}} = s^2 |a_{j_1 k_1}^w| s^{\frac{j_1+k_1}{2}}$$

for some $j_1, k_1 \geq 1$. Similarly, for each fixed n (and s),

$$(12) \quad \sum_{j, k \geq 1} |a_{jk}^w|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} \sim \max |a_{jk}^w|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} = |a_{j_0 k_0}^w|^n s^{n \frac{j_0+k_0}{2}} n^{n \frac{j_0+k_0}{2}}$$

for some choice of index $j_0, k_0 \geq 1$ (which depends on n and s). From (11), we have $|a_{j_1 k_1}^w| s^{\frac{j_1+k_1}{2}} \geq |a_{j_0 k_0}^w| s^{\frac{j_0+k_0}{2}}$. This inequality, together with (11) and (12) yield

$$\begin{aligned}
\frac{1}{s B_d(w, \sqrt{s})} \sum_{j, k \geq 1} |a_{jk}^w|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} &\sim \frac{(|a_{j_0 k_0}^w| s^{\frac{j_0+k_0}{2}})^n n^{n \frac{j_0+k_0}{2}}}{s^2 |a_{j_1 k_1}^w| s^{\frac{j_1+k_1}{2}}} \\
&\leq \frac{(|a_{j_0 k_0}^w| s^{\frac{j_0+k_0}{2}})^{n-1} n^{n \frac{j_0+k_0}{2}}}{s^2} \\
&\leq \frac{1}{s^2} \sum_{j, k \geq 1} |a_{jk}^w|^{n-1} s^{(n-1) \frac{j+k}{2}} n^{n \frac{j+k}{2}}.
\end{aligned}$$

This establishes that the first term (up a multiplicative constant) is larger than the second term in (10). To show that the second term is (up to a multiplicative constant) larger than the third term in (10), we observe that

$$\begin{aligned}
\frac{1}{sB_d(w, \sqrt{s})} \sum_{j,k \geq 1} |a_{jk}^w|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} &\gtrsim \frac{(\max\{|a_{jk}^w| s^{\frac{j+k}{2}} n^{\frac{j+k}{2}}\})^n}{s^2 \max\{|a_{jk}^w| s^{\frac{j+k}{2}} n^{\frac{j+k}{2}}\}} \\
&\sim \frac{1}{s^2} \sum_{j,k \geq 1} |a_{jk}^w|^{n-1} s^{(n-1) \frac{j+k}{2}} n^{(n-1) \frac{j+k}{2}} \\
&\geq \frac{1}{s^2 n^{\frac{1}{2} \deg p}} \sum_{j,k \geq 1} |a_{jk}^w|^{n-1} s^{(n-1) \frac{j+k}{2}} n^{n \frac{j+k}{2}}.
\end{aligned}$$

This establishes (10).

Thus, to show that $|\frac{\partial \mathcal{H}_{\tau p}}{\partial s}(s, z, w, t)|$ satisfies (6), with $J = J' = 0$, we will show the equivalent condition that there exist constants $C, A > 0$ so that

$$(13) \quad \|(t + T(w, z))^n \varphi\|_{L^\infty(\mathbb{R})} \leq \|M_{\tau p}^n \hat{\varphi}\|_{L^1(\mathbb{R})} \leq \frac{CA^n}{s^2} \sum_{j,k \geq 1} |a_{jk}^w|^{n-1} s^{(n-1) \frac{j+k}{2}} n^{n \frac{j+k}{2}}.$$

5. ESTIMATES FOR $M_{\tau p}^n H_{\tau p}(s, z, w)$ AND THE PROOF OF THEOREM 2.1

Since $\square_{\tau p}$ is a self-adjoint operator in $L^2(\mathbb{C})$, it follows that $H_{\tau p}(s, z, w) = \overline{H_{\tau p}(s, w, z)}$ [Rai06a]. Thus, the differential operators in w are:

$$\overline{W}_{\tau p, w} = \overline{(Z_{\tau p, w})} = \frac{\partial}{\partial \bar{w}} - \tau \frac{\partial p}{\partial \bar{w}} = e^{\tau p} \frac{\partial p}{\partial \bar{w}} e^{-\tau p}, \quad W_{\tau p, w} = \overline{(\overline{Z}_{\tau p, w})} = \frac{\partial}{\partial w} + \tau \frac{\partial p}{\partial w} = e^{-\tau p} \frac{\partial p}{\partial w} e^{\tau p}.$$

The goal of the remainder of the paper is to show the following theorem.

Theorem 5.1. *Let $p : \mathbb{C} \rightarrow \mathbb{R}$ be a subharmonic, nonharmonic polynomial, $\tau > 0$, and $n \geq 0$. Let c_0 be as in (24). There exists constants $C > 0$ so that*

(i)

$$|(M_{\tau p}^{z, w})^n H_{\tau p}(s, z, w)| \leq \begin{cases} \frac{C^n}{s \tau^2} e^{-c_0 \frac{|z-w|^2}{2s}} \sum_{j,k \geq 1} |a_{jk}^z|^{n-2} s^{(n-2) \frac{j+k}{2}} n^{n \frac{j+k}{2}} \\ \frac{C^n}{s} e^{-c_0 \frac{|z-w|^2}{2s}} \sum_{j,k \geq 1} |a_{jk}^z|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} \end{cases}$$

(ii) *If $X = \overline{Z}_{\tau p, z}, Z_{\tau p, z}, \overline{W}_{\tau p, w}$, or $W_{\tau p, w}$, then*

$$|X(M_{\tau p}^{z, w})^n H_{\tau p}(s, z, w)| \leq \begin{cases} \frac{C^n}{s^{3/2} \tau^2} e^{-c_0 \frac{|z-w|^2}{2s}} \sum_{j,k \geq 1} |a_{jk}^z|^{n-2} s^{(n-2) \frac{j+k}{2}} n^{n \frac{j+k}{2}} \\ \frac{C^n}{s^{3/2}} e^{-c_0 \frac{|z-w|^2}{2s}} \sum_{j,k \geq 1} |a_{jk}^z|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} \end{cases}$$

(iii) If $X^2 = W_{\tau p, w} \overline{W}_{\tau p, w}$ or $X = Z_{\tau p, z} \overline{W}_{\tau p, w}$, then

$$|X^2(M_{\tau p}^{z, w})^n H_{\tau p}(s, z, w)| \leq \begin{cases} \frac{C^n}{s^2 \tau^2} e^{-c_0 \frac{|z-w|^2}{4s}} \sum_{j, k \geq 1} |a_{jk}^z|^{n-2} s^{(n-2)\frac{j+k}{2}} n^{n\frac{j+k}{2}} \\ \frac{C^n}{s^2} e^{-c_0 \frac{|z-w|^2}{4s}} \sum_{j, k \geq 1} |a_{jk}^z|^n s^{n\frac{j+k}{2}} n^{n\frac{j+k}{2}} \end{cases}$$

Remark 5.2. The argument we give assumes $n \geq 3$. However, the $n \leq 2$ case follows from [Rai]. While the bounds in [Rai] have better decay in s and $|z - w|$ than in Theorem 5.1, the constants depend on n in an unknown way, hence we need the more careful argument presented here.

Also, observe that

$$n^{n\frac{j+k}{2}} \leq A^n (n-1)^{(n-1)\frac{j+k}{2}},$$

for a suitable constant A . This means that we have flexibility in the statement of Theorem 5.1 in the sense that $(n-2)^{n-2}$ could be replaced by n^n (or $(n-1)^{n-1}$), etc.

Remark 5.3. One trick that we use repeatedly is the fact that for any $\epsilon > 0$ and $n \geq 0$, there exists a constant $C_{\epsilon, n}$ so that

$$(14) \quad e^{-c\frac{a}{b}} \leq C_{\epsilon, n} e^{-(1-\epsilon)c\frac{a}{b}} \frac{b^n}{a^n}.$$

We will use this inequality by either commenting we may need to decrease c for a subsequent inequality to hold true or we may simply and mysteriously halve the constant in the exponential.

Theorem 5.1 allows us to prove Theorem 2.1.

Proof of Theorem 2.1. The reason that we estimate $|(M_{\tau p}^{z, w})^n \frac{\partial H_{\tau p}}{\partial s}(s, z, w)|$ first is that we can reduce the integral to the case when $\tau > 0$. To see how this works, we recall an observation from [Rai06a]. Since $\square_{\tau p}^k \overline{Z}_{\tau p} = \overline{Z}_{\tau p} \tilde{\square}_{\tau p}^k$ for all $k \geq 0$, it follows that

$$e^{-s \square_{\tau p}} \overline{Z}_{\tau p} = \overline{Z}_{\tau p} e^{-s \tilde{\square}_{\tau p}}.$$

On the kernel side, if dw is Lebesgue measure on $\mathbb{R}^2 = \mathbb{C}$, then

$$e^{-s \square_{\tau p}} \overline{Z}_{\tau p} \varphi(z) = \int_{\mathbb{C}} H_{\tau p}(s, z, w) \overline{Z}_{\tau p, w} \varphi(w) dw = - \int_{\mathbb{C}} \overline{W}_{\tau p, w} H_{\tau p}(s, z, w) \varphi(w) dw$$

and

$$\overline{Z}_{\tau p, z} e^{-s \tilde{\square}_{\tau p}} \varphi(z) = \int_{\mathbb{C}} \overline{Z}_{\tau p, z} \tilde{H}_{\tau p}(s, z, w) \varphi(w) dw.$$

Thus,

$$(15) \quad -\overline{W}_{\tau p, w} H_{\tau p}(s, z, w) = \overline{Z}_{\tau p, z} \tilde{H}_{\tau p}(s, z, w).$$

Since $M_{-\tau p}^{z, w} = \overline{M_{\tau p}^{z, w}}$, by (4) and (15), we have (for $\tau > 0$),

$$(16) \quad \overline{(M_{-\tau p}^{z, w})^n \frac{\partial H_{-\tau p}}{\partial s}(s, z, w)} = (M_{\tau p}^{z, w})^n \frac{\partial}{\partial s} \tilde{H}_{\tau p}(s, z, w) = (M_{\tau p}^{z, w})^n Z_{\tau p, z} \overline{Z}_{\tau p, z} \tilde{H}_{\tau p}(s, z, w) \\ = -(M_{\tau p}^{z, w})^n Z_{\tau p, z} \overline{W}_{\tau p, w} H_{\tau p}(s, z, w)$$

As a consequence of (16), we have successfully reduced to the estimate on $|(M_{\tau p}^{z,w})^n \frac{\partial H_{\tau p}}{\partial s}(s, z, w)|$ for $\tau \in \mathbb{R}$ to an estimate on $|(M_{\tau p}^{z,w})^n \frac{\partial H_{\tau p}}{\partial s}(s, z, w)|$ for $\tau > 0$.

With X^2 as in (iii), we need to show that we can estimate of $(M_{\tau p}^{z,w})^n X^2 H_{\tau p}(s, z, w)$ using Theorem 5.1. We handle one derivative at a time. Assume that X as in (ii) of Theorem 5.1. Let $e(w, z) = \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} p(z)}{\partial z^j \partial \bar{z}} (w - z)^j$. From Proposition 5.6 in [Rai],

$$e(w, z) = - \sum_{\substack{j \geq 1 \\ k \geq 0}} \frac{1}{j!k!} \frac{\partial^{j+k+1} p(w)}{\partial w^j \partial \bar{w}^{k+1}} (z - w)^j \overline{(z - w)}^k.$$

and

$$(17) \quad e(w, z) = -[M_{\tau p}, \overline{Z}_{\tau p}].$$

We can write

$$\begin{aligned} |(M_{\tau p}^{z,w})^n X H_{\tau p}(s, z, w)| &\leq |X(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)| + \sum_{j=0}^{n-1} |(M_{\tau p}^{z,w})^{n-1-j} [M_{\tau p}^{z,w}, X] (M_{\tau p}^{z,w})^j H_{\tau p}(s, z, w)| \\ &= |X(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)| + n |e(w, z)| |(M_{\tau p}^{z,w})^{n-1} H_{\tau p}(s, z, w)| \end{aligned}$$

Certainly, the only term to estimate is $|e(w, z)| |(M_{\tau p}^{z,w})^{n-1} H_{\tau p}(s, z, w)|$. Using (i), we have

$$\begin{aligned} &|e(w, z)| |(M_{\tau p}^{z,w})^{n-1} H_{\tau p}(s, z, w)| \\ &\leq \sum_{\alpha, \beta \geq 1} |a_{\alpha\beta}^z| |w - z|^{\alpha+\beta-1} e^{-c_0 \frac{|z-w|^2}{2s}} \frac{C^{n-1}}{s} \sum_{j,k \geq 1} |a_{jk}^z|^{n-1} s^{(n-1)\frac{j+k}{2}} (n-1)^{(n-1)\frac{j+k}{2}} \\ &\leq e^{-c_0 \frac{|z-w|^2}{4s}} \sum_{\alpha, \beta \geq 1} |a_{\alpha\beta}^z| s^{\frac{\alpha+\beta-1}{2}} \frac{C^{n-1}}{s} \sum_{j,k \geq 1} |a_{jk}^z|^{n-1} s^{(n-1)\frac{j+k}{2}} (n-1)^{(n-1)\frac{j+k}{2}} \\ &\leq e^{-c_0 \frac{|z-w|^2}{4s}} \frac{C^{n-1}}{s^{3/2}} \sum_{j,k \geq 1} |a_{jk}^z|^n s^{n\frac{j+k}{2}} n^{n\frac{j+k}{2}}. \end{aligned}$$

Thus $(M_{\tau p}^{z,w})^n X H_{\tau p}(s, z, w)$ satisfies the estimate (ii) in Theorem 5.1 for some uniform constant c_0 . By similar arguments, we can show that if X^2 is as in (iii) of Theorem 5.1, then $(M_{\tau p}^{z,w})^n X^2 H_{\tau p}(s, z, w)$ satisfies the estimates given in (iii) of Theorem 5.1 when $\tau > 0$ by cutting c_0 in half (again). Next, since $\frac{\partial H_{\tau p}}{\partial s}(s, z, w) = -W_{\tau p, w} \overline{W}_{\tau p, w} H_{\tau p}(s, z, w)$, it follows from the previous paragraph that $(M_{\tau p}^{z,w})^n \frac{\partial H_{\tau p}}{\partial s}(s, z, w)$ satisfies the estimates in (i) of Theorem 5.1 for all τ , both positive and negative (up to a modification of c_0).

Now we integrate this estimate in τ . Observe that

$$\int_0^\infty \min\{|a_{jk}^z|^2 s^{2\frac{j+k}{2}}, \tau^{-2}\} d\tau = \int_0^{(|a_{jk}^z| s^{\frac{j+k}{2}})^{-1}} |a_{jk}^z|^2 s^{j+k} d\tau + \int_{(|a_{jk}^z| s^{\frac{j+k}{2}})^{-1}}^\infty \tau^{-2} d\tau = 2|a_{jk}^z| s^{\frac{j+k}{2}}$$

Using this together with the estimate for $(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)$ in part (i) of Theorem 5.1 (for all τ), we have

$$\int_{-\infty}^\infty |(M_{\tau p}^{z,w})^n \frac{\partial H_{\tau p}}{\partial s}(s, z, w)| d\tau \leq \frac{C^n}{s^2} e^{-c_0 \frac{|z-w|^2}{4s}} \sum_{j,k \geq 1} |a_{jk}^z|^{n-1} s^{(n-1)\frac{j+k}{2}} n^{n\frac{j+k}{2}}.$$

By (13), this proves the following estimate:

$$(18) \quad \left| \frac{\partial}{\partial s} \mathcal{H}_{\tau p}(s, \alpha, \beta) \right| \leq C \frac{e^{-c \frac{d(\alpha, \beta)^2}{s}}}{s |B_d(\alpha, \sqrt{s})|}.$$

To recover the estimate for $\mathcal{H}_{\tau p}(s, \alpha, \beta)$, we use the Fundamental Theorem of Calculus and the fact that $\mathcal{H}_{\tau p}(0, \alpha, \beta) = 0$ away from the diagonal. If we set $d = d(\alpha, \beta)$ and consider the case $s \geq \frac{1}{2}c_0 d^2$, then we estimate (with replacing c_0 by a smaller constant c using (14)),

$$\begin{aligned} |\mathcal{H}_{\tau p}(s, \alpha, \beta)| &= \left| \int_0^s \frac{\partial \mathcal{H}_{\tau p}}{\partial s}(r, \alpha, \beta) dr \right| \leq \int_0^{d^2} \frac{C}{d^2(d^2 \Lambda(z, d))} e^{-c \frac{d^2}{s}} dr + \int_{d^2}^s \frac{C}{r^2 \Lambda(z, d)} e^{-c \frac{d^2}{s}} dr \\ &\leq \frac{C}{V(\alpha, \beta)} e^{-c \frac{d(\alpha, \beta)^2}{s}}. \end{aligned}$$

If $s \leq \frac{1}{2}c_0 d^2$, then we estimate (with c_0 replaced by the smaller c using (14)) that

$$|\mathcal{H}_{\tau p}(s, \alpha, \beta)| = \left| \int_0^s \frac{\partial \mathcal{H}_{\tau p}}{\partial s}(r, \alpha, \beta) dr \right| \leq \int_0^s \frac{C}{r V(\alpha, \beta)} e^{-c \frac{d^2}{2r}} dr.$$

If we set $f(r) = \frac{1}{r} e^{-c \frac{d^2}{2r}}$, then calculus shows that $f'(r) \geq 0$ when $r \leq \frac{1}{2}c_0 d^2$. This means

$$|\mathcal{H}_{\tau p}(s, \alpha, \beta)| \leq \frac{C}{V(\alpha, \beta)} \int_0^s f(r) dr \leq \frac{C}{V(\alpha, \beta)} s f(s) = \frac{C}{V(\alpha, \beta)} e^{-c_0 \frac{d^2}{2s}}.$$

The passage from estimates on $\mathcal{H}_{\tau p}(s, \alpha, \beta)$ to estimates on $X_\alpha^J X_\beta^{J'} \mathcal{H}_{\tau p}(s, \alpha, \beta)$ involves a short bootstrapping argument and Theorem 3.4.2 from [NS01a], a Sobolev embedding theorem. Fix $s > 0$ and $\beta \in \mathbb{C} \times \mathbb{R}$. We first bound derivatives only in α . From [NS01b], there exists a bump function $\varphi \in C_c^\infty(B_d(\alpha, \frac{1}{2}d(\alpha, \beta)))$ so that $\varphi(\gamma) = 1$ on $B_d(\alpha, \frac{1}{4}d(\alpha, \beta))$, $0 \leq \varphi \leq 1$ and for every multiindex I , $|X^I \varphi| \leq \frac{c_{|I|}}{d(\alpha, \beta)^{|I|}}$ where $c_{|I|}$ is independent of α and $d(\alpha, \beta)$. We now use Theorem 3.4.2 from [NS01a] (and note that we may take $R_0 = \infty$) and estimate that for some $C > 0$ and $L \in \mathbb{N}$,

$$(19) \quad \begin{aligned} |X_\alpha^I \mathcal{H}_{\tau p}(s, \alpha, \beta)| &= |\varphi(\alpha) X_\alpha^I \mathcal{H}_{\tau p}(s, \alpha, \beta)| \\ &\leq \frac{C}{V(\alpha, \beta)^{1/2}} \sum_{0 \leq |J| \leq L} d(\alpha, \beta)^{|J|} \|X_\alpha^J (\varphi X_\alpha^I \mathcal{H}_{\tau p}(s, \cdot, \beta))\|_{L^2(\mathbb{C} \times \mathbb{R})}. \end{aligned}$$

The derivatives in this estimation are taken with respect to α and we will henceforth omit the subscript. We integrate by parts using the fact that $(X^0)^* = -X^1$ (and $(X^1)^* = -X^0$) and obtain

$$(20) \quad \begin{aligned} \|X^J (\varphi X^I \mathcal{H}_{\tau p})\|_{L^2}^2 &= \langle X^J (\varphi X^I \mathcal{H}_{\tau p}), X^J (\varphi X^I \mathcal{H}_{\tau p}) \rangle = \langle \mathcal{H}_{\tau p}, (X^I)^* (\varphi (X^J)^* X^J (\varphi X^I \mathcal{H}_{\tau p})) \rangle \\ &\leq \|\mathcal{H}_{\tau p}\|_{L^\infty(\text{supp } \varphi)} V(\alpha, \beta)^{1/2} \|(X^I)^* (\varphi (X^J)^* X^J (\varphi X^I \mathcal{H}_{\tau p}))\|_{L^2}. \end{aligned}$$

Since $d(\gamma, \beta) \geq \frac{1}{2}d(\alpha, \beta)$ for $\gamma \in \text{supp } \varphi$,

$$|(X^I)^* (\varphi (X^J)^* X^J (\varphi X^I \mathcal{H}_{\tau p}))| \leq C \sum_{|I_1|+|I_2|+|I_3|=2|I|+2|J|} |X^{I_1} \varphi| |X^{I_2} \varphi| |X_\gamma^{I_3} \mathcal{H}_{\tau p}(s, \gamma, \beta)|,$$

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and from [NS01a] or [Rai], $|X_\gamma^{I_3} \mathcal{H}_{\tau p}(s, \gamma, \beta)| \leq C_{|I_3|} d(\gamma, \beta)^{-|I_3|} V(\gamma, \beta)^{-1}$, it follows that

$$\begin{aligned} \|(X^I)^*(\varphi(X^J)^* X^J(\varphi X^I \mathcal{H}_{\tau p}))\|_{L^2} &\leq \|(X^I)^*(\varphi(X^J)^* X^J(\varphi X^I \mathcal{H}_{\tau p}))\|_{L^\infty} V(\alpha, \beta)^{1/2} \\ &\leq \frac{C}{d(\alpha, \beta)^{2|I|+2|J|} V(\alpha, \beta)^{1/2}}. \end{aligned}$$

Using the estimate on $\mathcal{H}_{\tau p}(s, \gamma, \beta)$ proven above, we have that on $\text{supp } \varphi$, $|\mathcal{H}_{\tau p}(s, \gamma, \beta)| \leq C \frac{e^{-c \frac{d(\alpha, \beta)^2}{2s}}}{V(\alpha, \beta)}$, so plugging our estimate on $\|X^I(\varphi X^J X^J(\varphi X^I \mathcal{H}_{\tau p}))\|_{L^2}$ into (20) and that into (19), we get (with a further decrease in c) that

$$\begin{aligned} |X_\alpha^I \mathcal{H}_{\tau p}(s, \alpha, \beta)| &\leq \frac{C}{V(\alpha, \beta)^{1/2}} \sum_{0 \leq |J| \leq L} d(\alpha, \beta)^{|J|} \frac{e^{-c \frac{d(\alpha, \beta)^2}{s}}}{V(\alpha, \beta)^{1/2} V(\alpha, \beta)^{1/4}} \frac{1}{d(\alpha, \beta)^{|I|+|J|}} V(\alpha, \beta)^{1/4} \\ &\leq \frac{C}{d(\alpha, \beta)^{|I|}} \frac{e^{-c \frac{d(\alpha, \beta)^2}{s}}}{V(\alpha, \beta)}, \end{aligned}$$

the desired estimate. To pass from estimates on $X_\alpha^J \mathcal{H}_{\tau p}(s, \alpha, \beta)$ to estimates on $X_\alpha^J X_\beta^{J'} \mathcal{H}_{\tau p}(s, \alpha, \beta)$, we simply repeat the argument in β with $X_\alpha^J \mathcal{H}_{\tau p}(s, \alpha, \beta)$ playing the role of $\mathcal{H}_{\tau p}(s, \alpha, \beta)$. Finally, since $\frac{\partial^j}{\partial s^j} \mathcal{H}_{\tau p}(s, \alpha, \beta) = (-1)^j \square_b^j \mathcal{H}_{\tau p}(s, \alpha, \beta)$, proving the estimates for $X_\alpha^J X_\beta^{J'} \mathcal{H}_{\tau p}(s, \alpha, \beta)$ is sufficient to prove the theorem. \square

Remark 5.4. The estimates in Theorem 5.1 allow us to prove that $e^{-i\tau T(w, z)} H_{\tau p}(s, z, w)$ satisfies L^1 -QSE for every $\beta = j + k$ where $j \geq 1$, $k \geq 1$ (of course, for $j + k \geq \deg p$, the condition is vacuous). The exponential decay for $H(s, \alpha, \beta)$ follows by proving the L^1 -QSE and keeping careful track of the powers of s and $|a_{jk}^0|$.

6. A GOOD FORMULA FOR $M_{\tau p}^{z, w} H_{\tau p}(s, z, w)$

The goal of this section is to prove a tractable formula for $M_{\tau p}^{z, w} H_{\tau p}(s, z, w)$. The launching point is the solution to the nonhomogeneous heat equation in [Rai] given by a Duhamel's formula.

Proposition 5.1 in [Rai] yields

Proposition 6.1. *Let $g : (0, \infty) \times \mathbb{C} \rightarrow \mathbb{C}$ and $f : \mathbb{C} \rightarrow \mathbb{C}$ be $L^2(\mathbb{C})$ for each s and vanish as $|z| \rightarrow \infty$. The solution to the nonhomogeneous heat equation*

$$(21) \quad \begin{cases} \frac{\partial u}{\partial s} + \square_{\tau p} u = g & \text{in } (0, \infty) \times \mathbb{C} \\ \lim_{s \rightarrow 0} u(s, z) = f(z) \end{cases}$$

is given by

$$u(s, z) = \int_{\mathbb{C}} H_{\tau p}(s, z, \xi) f(\xi) d\xi + \int_0^s \int_{\mathbb{C}} H_{\tau p}(s - r, z, \xi) g(r, \xi) d\xi dr.$$

Observe that $(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)$ behaves as follows.

$$\begin{aligned}
g &:= \left(\frac{\partial}{\partial s} + \square_{\tau p, z} \right) (M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w) = M_{\tau p}^n \frac{\partial H_{\tau p}}{\partial s} + M_{\tau p} \square_{\tau p} M_{\tau p}^{n-1} H_{\tau p} + [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-1} H_{\tau p} \\
&= M_{\tau p}^n \frac{\partial H_{\tau p}}{\partial s} + M_{\tau p}^2 \square_{\tau p} M_{\tau p}^{n-2} H_{\tau p} + M_{\tau p} [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-2} H_{\tau p} + [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-1} H_{\tau p} \\
&= \cdots = \underbrace{M_{\tau p}^n \frac{\partial H_{\tau p}}{\partial s} + M_{\tau p}^n \square_{\tau p} H_{\tau p}}_{=0} + M_{\tau p}^{n-1} [\square_{\tau p}, M_{\tau p}] H_{\tau p} + M_{\tau p}^{n-2} [\square_{\tau p}, M_{\tau p}] M_{\tau p} H_{\tau p} \\
&\quad + \cdots M_{\tau p} [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-2} + [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-1} H_{\tau p}.
\end{aligned}$$

From [Rai], Proposition 5.4 we have

$$\begin{aligned}
(22) \quad [\square_{\tau p}, M_{\tau p}^{z,w}] &= \overline{Z_{\tau p, z} e(w, z)} - e(w, z) Z_{\tau p, z} = \overline{Z_{\tau p, z} e(w, z)} - Z_{\tau p, z} e(w, z) + \frac{\partial^2 p(z)}{\partial z \partial \bar{z}} \\
&= -\frac{\partial^2 p}{\partial z \partial \bar{z}} - e(w, z) Z_{\tau p, z} + \overline{e(w, z)} \overline{Z_{\tau p, z}}.
\end{aligned}$$

To simplify the calculation further, observe

$$\begin{aligned}
M_{\tau p} [\square_{\tau p}, M_{\tau p}] &= [\square_{\tau p}, M_{\tau p}] M_{\tau p} + [M_{\tau p}, [\square_{\tau p}, M_{\tau p}]] \\
&= [\square_{\tau p}, M_{\tau p}] M_{\tau p} + [M_{\tau p}, -\frac{\partial^2 p}{\partial z \partial \bar{z}} - e(w, z) Z_{\tau p, z} + \overline{e(w, z)} \overline{Z_{\tau p, z}}] \\
&= [\square_{\tau p}, M_{\tau p}] M_{\tau p} - e(w, z) [M_{\tau p}, Z_{\tau p}] + \overline{e(w, z)} [M_{\tau p}, \overline{Z_{\tau p}}] \\
&= [\square_{\tau p}, M_{\tau p}] M_{\tau p} - 2|e(w, z)|^2.
\end{aligned}$$

where the next to last equality uses (17). Consequently,

$$\begin{aligned}
M_{\tau p}^j [\square_{\tau p}, M_{\tau p}] &= M_{\tau p}^{j-1} (M_{\tau p} [\square_{\tau p}, M_{\tau p}]) = M_{\tau p}^{j-1} ([\square_{\tau p}, M_{\tau p}] M_{\tau p} - 2|e(w, z)|^2) \\
&= M_{\tau p}^{j-2} [\square_{\tau p}, M_{\tau p}] M_{\tau p}^2 - 2M_{\tau p}^{j-2} |e(w, z)|^2 M_{\tau p} - 2M_{\tau p}^{j-1} |e(w, z)|^2 \\
&= M_{\tau p}^{j-2} [\square_{\tau p}, M_{\tau p}] M_{\tau p}^2 - 4|e(w, z)|^2 M_{\tau p}^{j-1} \\
&= \cdots = [\square_{\tau p}, M_{\tau p}] M_{\tau p}^j - 2j|e(w, z)|^2 M_{\tau p}^{j-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g &= \sum_{j=0}^{n-1} M_{\tau p}^j [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-1-j} H_{\tau p} = \sum_{j=0}^{n-1} \left([\square_{\tau p}, M_{\tau p}] M_{\tau p}^j - 2j|e(w, z)|^2 M_{\tau p}^{j-1} \right) M_{\tau p}^{n-1-j} H_{\tau p} \\
&= \sum_{j=0}^{n-1} [\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-1} H_{\tau p} - |e(w, z)|^2 M_{\tau p}^{n-2} H_{\tau p} \sum_{j=1}^{n-1} 2j \\
&= n[\square_{\tau p}, M_{\tau p}] M_{\tau p}^{n-1} H_{\tau p} - n(n-1)|e(w, z)|^2 M_{\tau p}^{n-2} H_{\tau p}.
\end{aligned}$$

From Theorem 6.3 in [Rai], it follows that the single integral term in Proposition 6.1 is 0, so we have:

Proposition 6.2.

$$(23) \quad (M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w) = n \int_0^s \int_{\mathbb{C}} H_{\tau p}(s-r, z, \xi) [\square_{\tau p, \xi}, M_{\tau p}^{\xi, w}] (M_{\tau p}^{\xi, w})^{n-1} H_{\tau p}(r, \xi, w) d\xi dr \\ - n(n-1) \int_0^s \int_{\mathbb{C}} H_{\tau p}(s-r, z, \xi) |e(w, \xi)|^2 (M_{\tau p}^{\xi, w})^{n-2} H_{\tau p}(r, \xi, w) d\xi dr.$$

We use Proposition 6.2 as a starting point for a recursion to generate a formula for $(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)$ that involves no τ -derivatives of $H_{\tau p}(s, z, w)$. Plugging the integral for $(M_{\tau p}^{z,w})^{n-1} H_{\tau p}(s, z, w)$ and $(M_{\tau p}^{z,w})^{n-2} H_{\tau p}(s, z, w)$ into the RHS of (23), we have

$$(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w) \\ = n(n-1) \int_0^s \int_0^{r_1} \int_{\mathbb{C}^2} H_{\tau p}(s-r_1, z, \xi_1) [\square_{\tau p, \xi_1}, M_{\tau p}^{\xi_1, w}] H_{\tau p}(r_1-r_2, \xi_1, \xi_2) \\ \times [\square_{\tau p, \xi_2}, M_{\tau p}^{\xi_2, w}] (M_{\tau p}^{\xi_2, w})^{n-2} H_{\tau p}(r_2, \xi_2, w) d\xi_2 d\xi_1 dr_2 dr_1 \\ - n(n-1)(n-2) \int_0^s \int_0^{r_1} \int_{\mathbb{C}^2} H_{\tau p}(s-r_1, z, \xi_1) [\square_{\tau p, \xi_1}, M_{\tau p}^{\xi_1, w}] H_{\tau p}(r_1-r_2, \xi_1, \xi_2) \\ \times |e(w, \xi_2)|^2 (M_{\tau p}^{\xi_2, w})^{n-3} H_{\tau p}(r_2, \xi_2, w) d\xi_2 d\xi_1 dr_2 dr_1 \\ - n(n-1)(n-2) \int_0^s \int_0^{r_1} \int_{\mathbb{C}^2} H_{\tau p}(s-r_1, z, \xi_1) |e(w, \xi_1)|^2 H_{\tau p}(r_1-r_2, \xi_1, \xi_2) \\ \times [\square_{\tau p, \xi_1}, M_{\tau p}^{\xi_2, w}] (M_{\tau p}^{\xi_2, w})^{n-3} H_{\tau p}(r_2, \xi_2, w) d\xi_2 d\xi_1 dr_2 dr_1 \\ + n(n-1)(n-2)(n-3) \int_0^s \int_0^{r_1} \int_{\mathbb{C}^2} H_{\tau p}(s-r_1, z, \xi_1) |e(w, \xi_1)|^2 H_{\tau p}(r_1-r_2, \xi_1, \xi_2) \\ \times |e(w, \xi_2)|^2 (M_{\tau p}^{\xi_2, w})^{n-4} H_{\tau p}(r_2, \xi_2, w) d\xi_2 d\xi_1 dr_2 dr_1$$

The procedure is repeated while there are still $M_{\tau p} H_{\tau p}$ terms left in the integrals. To calculate the resulting integral, a number of observations are needed. First, since the integral for n -derivatives decomposes to a sum involving $(n-1)$ -derivatives and $(n-2)$ -derivatives, if f_n is the number of integrals that n -derivatives decomposes into, then we have the relation

$$f_n = f_{n-1} + f_{n-2}.$$

Also, we know that $f_1 = 1$ and by Proposition 6.2, $f_2 = 2$. Thus, f_n is the n th Fibonacci number and

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right).$$

The important feature of f_n is that it grows geometrically with n (and not faster!). It is easiest to describe the derivation for the formula for $(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)$ in the language of trees. The descendants of $(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)$ are an integral that involves $(M_{\tau p}^{z,w})^{n-1} H_{\tau p}(s, z, w)$ and an integral that involves $(M_{\tau p}^{z,w})^{n-2} H_{\tau p}(s, z, w)$. The child that inherits the term with $(M_{\tau p}^{z,w})^{n-1} H_{\tau p}(s, z, w)$ comes with a factor n and the commutator $[\square_{\tau p}, M_{\tau p}]$. The child with $(M_{\tau p}^{z,w})^{n-2} H_{\tau p}(s, z, w)$ inherits a factor of $-n(n-1)$ and an $|e(w, \xi)|^2$ -term. We know that there are f_n paths down the tree. Let the left child denote the term where $M_{\tau p}$ drops by one degree and the right child denote the term where $M_{\tau p}$ drops by two degrees. Let \mathcal{I}_n denote the set of paths down tree for $(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)$. A path $J \in \mathcal{I}_n$ is a sequence $\{a_j\}$ with

$a_j = 1$ indicating a “left” child and $a_j = 2$ indicating a “right” child. The path length is $|J|$. It follows that $n/2 \leq |J| \leq n$. Let $J_1 = \#\{j \in J : a_j = 1\}$ and $J_2 = \#\{j \in J : a_j = 2\}$. Let

$$N(a_j, \xi_j) = \begin{cases} [\square_{\tau p, \xi_j}, M_{\tau p}^{\xi_j, w}] & a_j = 1 \\ |e(w, \xi_j)|^2 & a_j = 2 \end{cases}.$$

The operator $N(a_j, \xi_j)$ records the information discussed above. It follows that

Proposition 6.3.

$$\begin{aligned} & (M_{\tau p}^{z, w})^n H_{\tau p}(s, z, w) \\ &= n! \sum_{J \in \mathcal{I}_n} (-1)^{J_2} \int_0^s \int_0^{r_1} \cdots \int_0^{r_{|J|-1}} \int_{\mathbb{C}^{|J|}} H_{\tau p}(s - r_1, z, \xi_1) \left(\prod_{j=1}^{|J|-1} N(a_j, \xi_j) H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1}) \right) \\ & \times N(a_{|J|}, \xi_{|J|}) H_{\tau p}(r_{|J|}, \xi_{|J|}, w) d\xi_{|J|} \cdots d\xi_1 dr_{|J|} \cdots dr_1. \end{aligned}$$

7. PROOF OF THEOREM 5.1

Understanding how to manipulate the formula in Proposition 6.3 is the crux of the proof. The three parts of Theorem 5.1 are proven similarly, though not identically. We will start with (i) and prove it in detail. We will discuss the modifications necessary for (ii) and (iii). The workhorse estimates for proving Theorem 5.1 are the following estimates from [Rai06a]. When $\tau > 0$,

$$(24) \quad |H_{\tau p}(s, z, w)| \leq \frac{C}{s} e^{-c_0 \frac{|z-w|^2}{s}} e^{-c_0 \frac{s}{\mu(z, 1/\tau)^2}} e^{-c_0 \frac{s}{\mu(w, 1/\tau)^2}}.$$

and

$$(25) \quad |\overline{Z}_{\tau p, z} H_{\tau p}(s, z, w)| + |Z_{\tau p} H_{\tau p}(s, z, w)| \leq \frac{C}{s^{3/2}} e^{-c_0 \frac{|z-w|^2}{s}} e^{-c_0 \frac{s}{\mu(z, 1/\tau)^2}} e^{-c_0 \frac{s}{\mu(w, 1/\tau)^2}}.$$

Remark 7.1. When $\tau < 0$, $H_{\tau p}(s, z, w)$ satisfies a weaker estimate (proven in [Rai07]). Fortunately, we avoid this difficulty here by exploiting the equality $\square_{-\tau p} = \widetilde{\square}_{\tau p}$ and the fact that we can write certain derivatives of $\tilde{H}_{\tau p}(s, z, w)$ in terms of $H_{\tau p}(s, z, w)$ as done in (16).

Since there are only f_n -terms in the calculation and f_n grows geometrically with n , we can treat each integral from Proposition 6.3 separately. The integrals can all be handled analogously, and we choose to show a specific one for expositional clarity. We will show the case when $a_j = 1$ for all j . Even more specifically, $[\square_{\tau p}, M_{\tau p}]$, as given in the second line of (22), contains three terms. We concentrate on the term that always has $e(w, \xi) Z_{\tau p, \xi}$. Without loss of generality, we can take $w = 0$ since the argument is the same regardless of the w we choose. The integral we estimate is

$$\begin{aligned} I := & \left| \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} H_{\tau p}(s - r_1, z, \xi_1) \left(\prod_{j=1}^{n-1} e(0, \xi_j) Z_{\tau p, \xi_j} H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1}) \right) \right. \\ & \left. e(0, \xi_n) Z_{\tau p, \xi_n} H_{\tau p}(r_n, \xi_n, 0) d\xi_n \cdots d\xi_1 dr_n \cdots dr_1 \right|. \end{aligned}$$

The following inequality follows from the concavity of the logarithm and the convexity of x^k .

Lemma 7.2. *Let k be a positive integer and $a_1, \dots, a_k > 0$. Then*

$$(a_1 \cdots a_k) \leq \frac{1}{k}(a_1^k + \cdots + a_k^k).$$

The inequality is seen to be sharp by considering $a_1 = \cdots = a_k = a$. The other extremely useful fact is that

$$(26) \quad \exp\left(-c_0 \frac{|\xi_{k-1} - \xi_k|^2}{r_{k-1} - r_k}\right) \exp\left(-c_0 \frac{|\xi_k|^2}{r_k}\right) \\ = \exp\left(-c_0 \frac{r_{k-1}}{(r_{k-1} - r_k)r_k} \left|\xi_k - \frac{r_k}{r_{k-1}} \xi_{k-1}\right|^2\right) \exp\left(-c_0 \frac{|\xi_{k-1}|^2}{r_{k-1}}\right).$$

We now start the proof of the estimates of Theorem 5.1. First, we will handle the estimates without the term $1/\tau^2$ on the right; these will be referred to as the estimates without τ -decay. Then the argument will then be modified to establish the estimates with $1/\tau^2$ on the right, and these will be referred to as the estimates with τ -decay.

7.1. Estimate (i) of I without τ -decay. By Lemma 7.2, we have

$$|e(0, \xi_1) \cdots e(0, \xi_n)| \leq \frac{1}{n}(|e(0, \xi_1)|^n + \cdots + |e(0, \xi_n)|^n).$$

We let $C = C(p)$ (or A) be a constant that may vary from line to line and may depend on $\deg(\Delta p) + 2$ but NOT on n , the coefficients of p , or s . By (24) and (25), we have

$$I \leq \frac{A^n}{n} \int_0^s \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} \frac{e^{-c_0 \frac{|z - \xi_1|^2}{s - r_1}}}{s - r_1} \left(\frac{\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}}}{(r_m - r_{m+1})^{3/2}} \right) \frac{e^{-c_0 \frac{|\xi_n|^2}{r_n}}}{r_n^{3/2}} \\ (|e(0, \xi_1)|^n + \cdots + |e(0, \xi_n)|^n) d\xi_n \cdots d\xi_1 dr_n \cdots dr_1.$$

Note that we have ignored the terms on the right in (24) and (25) that involve decay in τ for this part of the argument. Choosing an arbitrary $e(0, \xi_\ell)$ term, we estimate the space integral first. Also, set $r_0 = s$.

$$\int_{\mathbb{C}^n} e^{-c_0 \frac{|z - \xi_1|^2}{s - r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |e(0, \xi_\ell)|^n d\xi_n \cdots d\xi_1 \\ \leq C \int_{\mathbb{C}^n} e^{-c_0 \frac{|z - \xi_1|^2}{s - r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} \left(\sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0| |\xi_\ell|^{j+k} \right)^n d\xi_n \cdots d\xi_1 \\ \leq A^n \sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0|^n \int_{\mathbb{C}^n} e^{-c_0 \frac{|z - \xi_1|^2}{s - r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |\xi_\ell|^{n(j+k)} d\xi_n \cdots d\xi_1 \\ \leq A^n e^{-\frac{c_0}{2} \frac{|z|^2}{s}} \sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0|^n \int_{\mathbb{C}^n} \left[\prod_{m=1}^{\ell} e^{-\frac{c_0}{2} \frac{r_{m-1}}{(r_{m-1} - r_m)r_m} |\xi_m - \frac{r_m}{r_{m-1}} \xi_{m-1}|^2} \right] \\ \left(\prod_{\alpha=\ell+1}^n e^{-c_0 \frac{r_{\alpha-1}}{(r_{\alpha-1} - r_\alpha)r_\alpha} |\xi_\alpha - \frac{r_\alpha}{r_{\alpha-1}} \xi_{\alpha-1}|^2} \right) e^{-\frac{c_0}{2} \frac{|\xi_\ell|^2}{r_\ell}} |\xi_\ell|^{n(j+k)} d\xi_n \cdots d\xi_1,$$

where the last inequality uses (26) repeatedly. By (7),

$$e^{-\frac{c_0}{2} \frac{|\xi_\ell|^2}{r_\ell}} |\xi_\ell|^{n(j+k)} \leq \left(\frac{(j+k)r_\ell}{ec_0} \right)^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}} \leq A^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}}$$

for all $1 \leq j+k \leq \deg(p)$. Consequently,

$$\begin{aligned} & \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m-\xi_{m+1}|^2}{r_m-r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |e(0, \xi_\ell)|^n d\xi_n \cdots d\xi_1 \\ & \leq A^n e^{-\frac{c_0}{2} \frac{|z|^2}{s}} \sum_{\substack{j \geq 1 \\ k \geq 0}} (|a_{j(k+1)}^0|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}}) \int_{\mathbb{C}^n} \left[\prod_{m=1}^{\ell} e^{-\frac{c_0}{2} \frac{r_{m-1}}{(r_{m-1}-r_m)r_m} |\xi_m - \frac{r_m}{r_{m-1}} \xi_{m-1}|^2} \right] \\ & \quad \left(\prod_{\alpha=\ell+1}^n e^{-c_0 \frac{r_{\alpha-1}}{(r_{\alpha-1}-r_\alpha)r_\alpha} |\xi_\alpha - \frac{r_\alpha}{r_{\alpha-1}} \xi_{\alpha-1}|^2} \right) d\xi_n \cdots d\xi_1 \\ & = A^n e^{-\frac{c_0}{2} \frac{|z|^2}{s}} \sum_{\substack{j \geq 1 \\ k \geq 0}} (|a_{j(k+1)}^0|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}}) \frac{(s-r_1)r_1}{s} \left(\prod_{m=1}^{n-1} \frac{(r_m-r_{m+1})r_{m+1}}{r_m} \right). \end{aligned}$$

Plugging this space integral estimate into the estimate for I , we have

$$\begin{aligned} I & \leq \frac{A^n}{sn} e^{-\frac{c_0}{2} \frac{|z|^2}{s}} \sum_{\substack{j \geq 1 \\ k \geq 0}} (|a_{j(k+1)}^0|^n s^{n \frac{j+k}{2}} n^{n \frac{j+k}{2}}) \\ & \quad \times \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-1}} (r_1-r_2)^{-1/2} \cdots (r_{n-1}-r_n)^{-1/2} r_n^{-1/2} dr_n \cdots dr_1. \end{aligned}$$

To estimate the convolutions in the time (i.e., r -integrals), we use the β -function result

$$(27) \quad \int_0^r \frac{s^{m/2-1}}{(r-s)^{1/2}} ds = r^{\frac{m+1}{2}-1} \int_0^1 s^{\frac{m}{2}-1} (1-s)^{\frac{1}{2}-1} ds = r^{\frac{m+1}{2}-1} \sqrt{\pi} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m+1}{2})}.$$

Thus,

$$\begin{aligned} & \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-1}} (r_1-r_2)^{1/2-1} \cdots (r_{n-1}-r_n)^{1/2-1} r_n^{1/2-1} dr_n \cdots dr_1 \\ & = \sqrt{\pi} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{2})} \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-2}} (r_1-r_2)^{1/2-1} \cdots (r_{n-2}-r_{n-1})^{1/2-1} r_{n-1}^{1-1} dr_{n-1} \cdots dr_1 \\ & = \pi^{2/2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{2})} \frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})} \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-3}} (r_1-r_2)^{1/2-1} \cdots (r_{n-3}-r_{n-2})^{1/2-1} r_{n-2}^{\frac{3}{2}-1} dr_{n-2} \cdots dr_1 \\ & = \cdots = \pi^{\frac{n-1}{2}} \frac{\Gamma(1/2)}{\Gamma(n/2)} \int_0^s r_1^{\frac{n}{2}-1} dr_1 = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})} s^{n/2}. \end{aligned}$$

Combining our estimates together, we have

$$I \leq \frac{A^n}{sn^2\Gamma(n/2)} e^{-\frac{c_0}{2} \frac{|z|^2}{s}} \sum_{\substack{j \geq 1 \\ k \geq 0}} (|a_{j(k+1)}^0|^n s^{n \frac{j+(k+1)}{2}} n^{n \frac{j+k}{2}}).$$

It turns out that the $n\Gamma(n/2)$ term is exactly what we need to attain (13). In the statement of Proposition 6.3, there is an $n!$ multiplying the integral. By Stirling's formula, we can bound

$$\frac{n!}{n^2\Gamma(n/2)} \leq A^n \frac{n^n}{n^{n/2}} = A^n n^{n/2}.$$

Therefore,

$$n^{n\frac{j+k}{2}} \frac{n!}{n^2\Gamma(n/2)} \leq A^n n^{n\frac{j+(k+1)}{2}}$$

Reindexing our sum and interchanging z and w , we have shown that for $\tau > 0$,

$$|(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)| \leq \frac{A^n}{s} e^{-c_0 \frac{|z|^2}{2s}} \sum_{j,k \geq 1} |a_{jk}^z|^n s^{n\frac{j+k}{2}} n^{n\frac{j+k}{2}}$$

which is the desired estimate in (i) without decay.

7.2. Second estimation of I with decay. This time we will exploit the τ decay terms in (24) and (25), (including those depending on $\mu(\xi, 1/\tau)$). We also apply Lemma 7.2 to $\prod_{j=1}^{n-2} |e(0, \xi_j)|$ leaving $|e(0, \xi_{n-1})||e(0, \xi_n)|$ alone. We obtain

$$\begin{aligned} I &\leq \frac{A^{n-2}}{(n-2)} \int_0^s \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} \frac{e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}}}{s-r_1} \left(\prod_{j=1}^{n-1} \frac{e^{-c_0 \frac{|\xi_j-\xi_{j+1}|^2}{r_j-r_{j+1}}}}{(r_j-r_{j+1})^{3/2}} \right) \frac{e^{-c_0 \frac{|\xi_n|^2}{r_n}}}{r_n^{3/2}} \\ &\quad \times (|e(0, \xi_1)|^{n-2} + \cdots + |e(0, \xi_{n-2})|^{n-2}) |e(0, \xi_{n-1})| |e(0, \xi_n)| \\ &\quad \times e^{-c_0 \frac{r_{n-2}-r_n}{\mu(\xi_{n-1}, 1/\tau)^2}} e^{-c_0 \frac{r_{n-1}}{\mu(\xi_n, 1/\tau)^2}} e^{-c_0 \frac{r_n}{\mu(0, 1/\tau)^2}} d\xi_n \cdots d\xi_1 dr_n \cdots dr_1. \end{aligned}$$

In the above calculation, we used $\exp(-c_0 \frac{r_{n-2}-r_{n-1}}{\mu(\xi_{n-1}, 1/\tau)}) \exp(-c_0 \frac{r_{n-1}-r_n}{\mu(\xi_{n-1}, 1/\tau)}) = \exp(-c_0 \frac{r_{n-2}-r_n}{\mu(\xi_{n-1}, 1/\tau)})$, which explains the appearance of this term in the above integrand.

We pick just one $|e(0, \xi_\ell)|^{n-2}$ term and concentrate on the space integral. Using (7), we have

$$e^{-c_0 \frac{|\xi_\ell|^2}{r_\ell}} |e(0, \xi_\ell)|^{n-2} \leq \sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0|^{n-2} s^{(n-2)\frac{j+k}{2}} (n-2)^{(n-2)\frac{j+k}{2}}$$

(since $r_\ell \leq s$). By a repeated use of (26) as we did in our first estimate of I , we have

$$\begin{aligned} II &:= \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left(\prod_{j=1}^{n-1} e^{-c_0 \frac{|\xi_j-\xi_{j+1}|^2}{r_j-r_{j+1}}} \right) |e(0, \xi_\ell)|^{n-2} e^{-c_0 \frac{|\xi_n|^2}{r_n}} e^{-c_0 \frac{r_{n-2}-r_n}{\mu(\xi_{n-1}, 1/\tau)^2}} \\ &\quad \times |e(0, \xi_{n-1})| |e(0, \xi_n)| e^{-c_0 \frac{r_{n-1}}{\mu(\xi_n, 1/\tau)^2}} e^{-c_0 \frac{r_n}{\mu(0, 1/\tau)^2}} d\xi_n \cdots d\xi_1 \\ &\leq A^n e^{-\frac{c_0}{2} \frac{|z|^2}{s}} \sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0|^{n-2} s^{(n-2)\frac{j+k}{2}} (n-2)^{(n-2)\frac{j+k}{2}} \int_{\mathbb{C}^n} \left[\prod_{m=1}^n e^{-\frac{c_0}{2} \frac{r_{m-1}}{(r_{m-1}-r_m)r_m}} |\xi_m - \frac{r_m}{r_{m-1}} \xi_{m-1}|^2 \right] \\ &\quad \times \left(|e(0, \xi_n)| e^{-\frac{c_0}{4} \frac{|\xi_n|^2}{r_n}} e^{-c_0 \frac{r_{n-1}}{\mu(\xi_n, 1/\tau)^2}} \left(|e(0, \xi_{n-1})| e^{-\frac{c_0}{4} \frac{|\xi_{n-1}|^2}{r_{n-1}}} e^{-c_0 \frac{r_n}{\mu(\xi_0, 1/\tau)^2}} e^{-c_0 \frac{r_{n-2}-r_n}{\mu(\xi_{n-1}, 1/\tau)^2}} \right) d\xi_n \cdots d\xi_1. \right. \end{aligned}$$

Again using (7), we have

$$\begin{aligned}
|e(0, \xi_n)| e^{-\frac{c_0}{4} \frac{|\xi_n|^2}{r_n}} e^{-c_0 \frac{r_{n-1}}{\mu(\xi_n, 1/\tau)^2}} &\leq C \sum_{j \geq 1} |a_{j1}^{\xi_n}| |\xi_n|^j \frac{r_n^{j/2} r_{n-1}^{1/2}}{|\xi_n|^j r_{n-1}^{1/2}} \frac{\mu(\xi_n, 1/\tau)^{j+1}}{r_{n-1}^{(j+1)/2}} \\
&= \frac{C}{r_{n-1}^{1/2}} \Lambda(\xi_n, \mu(\xi_n, 1/\tau)) \leq \frac{C}{\tau r_{n-1}^{1/2}}.
\end{aligned}$$

Since, $\max\{r_n, r_{n-2} - r_n\} \geq \frac{1}{2}r_{n-2}$, a similar argument shows that

$$|e(0, \xi_{n-1})| e^{-\frac{c_0}{4} \frac{|\xi_{n-1}|^2}{r_{n-1}}} e^{-c_0 \frac{r_n}{\mu(0, 1/\tau)^2}} e^{-c_0 \frac{r_{n-2} - r_n}{\mu(\xi_{n-1}, 1/\tau)^2}} \leq \frac{C}{\tau r_{n-2}^{1/2}}.$$

Consequently,

$$II \leq \frac{A^n e^{-c_0 \frac{|z|^2}{2s}}}{\tau^2} \sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0|^{n-2} s^{(n-2) \frac{j+k}{2}} (n-2)^{(n-2) \frac{j+k}{2}} \left(\prod_{m=1}^n \frac{(r_{m-1} - r_m) r_m}{r_{m-1}} \right) r_{n-2}^{-1/2} r_{n-1}^{-1/2}.$$

The time r -integrals become

$$\begin{aligned}
&\frac{1}{s} \int_0^s \cdots \int_0^{r_{n-1}} (r_1 - r_2)^{\frac{1}{2}-1} \cdots (r_{n-1} - r_n)^{\frac{1}{2}-1} r_{n-2}^{\frac{1}{2}-1} r_{n-1}^{\frac{1}{2}-1} r_n^{\frac{1}{2}-1} dr_n \cdots dr_1 \\
&= \frac{\pi^2}{s} \int_0^s \cdots \int_0^{r_{n-3}} (r_1 - r_2)^{\frac{1}{2}-1} \cdots (r_{n-3} - r_{n-2})^{\frac{1}{2}-1} r_{n-2}^{\frac{1}{2}-1} dr_{n-2} \cdots dr_1 \\
&= \frac{\pi^{2+\frac{n-2}{2}}}{s} \frac{1}{\frac{n-2}{2} \Gamma(\frac{n-2}{2})} s^{\frac{n-2}{2}}.
\end{aligned}$$

Thus using similar arguments to those at the end of the first estimate, we obtain

$$|(M_{\tau p}^{z,w})^n H_{\tau p}(s, z, w)| \leq \frac{A^n}{s \tau^2} e^{-c_0 \frac{|z|^2}{2s}} \sum_{j,k \geq 1} |a_{jk}^z|^{n-2} s^{(n-2) \frac{j+k}{2}} (n-2)^{(n-2) \frac{j+k}{2}}$$

which establishes (i) with decay.

7.3. Proof of Theorem 5.1, (iii) with no decay in τ . Let $r_{|J|+1} = 0$ and $\xi_{|J|+1} = w = 0$. The starting point for (iii) is Proposition 6.3. We consider the case when $X^2 = \overline{W}_{\tau p, w} Z_{\tau p, z}$

and outline the differences needed for other second derivative combinations later. We have

$$\begin{aligned}
& \overline{W}_{\tau p, w} Z_{\tau p, z} (M_{\tau p}^{z, w})^n H_{\tau p}(s, z, 0) \\
(28) \quad &= n! \sum_{J \in \mathcal{I}_n} (-1)^{J_2} \int_0^s \int_0^{r_1} \cdots \int_0^{r_{|J|-1}} \int_{\mathbb{C}^{|J|}} Z_{\tau p, z} H_{\tau p}(s - r_1, z, \xi_1) \left(\prod_{j=1}^{|J|-1} N(a_j, \xi_j) H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1}) \right) \\
&\quad \times N(a_{|J|}, \xi_{|J|}) \overline{W}_{\tau p, w} H_{\tau p}(r_{|J|}, \xi_{|J|}, 0) d\xi_{|J|} \cdots d\xi_1 dr_{|J|} \cdots dr_1 \\
(29) \quad &+ n! \sum_{k=1}^{|J|} \sum_{J \in \mathcal{I}_n} (-1)^{J_2} \int_0^s \int_0^{r_1} \cdots \int_0^{r_{|J|-1}} \int_{\mathbb{C}^{|J|}} Z_{\tau p, z} H_{\tau p}(s - r_1, z, \xi_1) \\
&\times \left(\prod_{\substack{j=1 \\ j \neq k}}^{|J|} N(a_j, \xi_j) H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1}) \right) \left(\frac{\partial}{\partial \bar{w}} N(a_k, \xi_k) \right) H_{\tau p}(r_k - r_{k+1}, \xi_k, \xi_{k+1}) d\xi_{|J|} \cdots d\xi_1 dr_{|J|} \cdots dr_1
\end{aligned}$$

The first integral is the most difficult to bound. We concentrate on that integral and mention at the end how to deal with integrals in the second sum.

The issue is the convergence of the time integrals. Each spacial derivative of $H_{\tau p}$ increases the power s (or $(r_j - r_{j+1})$) in the denominator by $1/2$, so we have to be careful in our estimation. The trick here is to use the $e(0, \xi_n)$ term. As above, we demonstrate the estimation on

$$\begin{aligned}
III := & \left| \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} Z_{\tau p, z} H_{\tau p}(s - r_1, z, \xi_1) \left(\prod_{j=1}^{n-1} e(0, \xi_j) Z_{\tau p, \xi_j} H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1}) \right) \right. \\
& \left. e(0, \xi_n) Z_{\tau p, \xi_n} \overline{W}_{\tau p, w} H_{\tau p}(r_n, \xi_n, 0) d\xi_n \cdots d\xi_1 dr_n \cdots dr_1 \right|.
\end{aligned}$$

Using (25) and Lemma 7.2, we have

$$\begin{aligned}
III \leq & \frac{A^n}{n-1} \int_0^s \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} \frac{e^{-c_0 \frac{|z - \xi_1|^2}{s - r_1}}}{(s - r_1)^{3/2}} \left(\prod_{m=1}^{n-1} \frac{e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}}}{(r_m - r_{m+1})^{3/2}} \right) \frac{e^{-c_0 \frac{|\xi_n|^2}{r_n}}}{r_n^2} \\
& |e(0, \xi_n)| (|e(0, \xi_1)|^{n-1} + \cdots + |e(0, \xi_{n-1})|^{n-1}) d\xi_n \cdots d\xi_1 dr_n \cdots dr_1.
\end{aligned}$$

As above, we concentrate on the space integral first. We estimate

$$\begin{aligned}
& \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |e(0, \xi_n)| |e(0, \xi_\ell)|^{n-1} d\xi_n \cdots d\xi_1 \\
& \leq C \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} \left(\sum_{\substack{j \geq 1 \\ k \geq 0}} |a_{j(k+1)}^0| |\xi_\ell|^{j+k} \right)^{n-1} |e(0, \xi_n)| d\xi_n \cdots d\xi_1 \\
& \leq C^n \int_{\mathbb{C}^n} e^{-\frac{c_0}{4} \frac{|z|^2}{s}} \left(\prod_{m=1}^n e^{-\frac{c_0}{4} \frac{r_{m-1}}{(r_{m-1} - r_m)r_m} |\xi_m - \frac{r_m}{r_{m-1}} \xi_{m-1}|^2} \right) \\
& \quad \times e^{-\frac{c_0}{4} \frac{|\xi_n|^2}{r_n}} \left(\sum_{j,k \geq 1} |a_{jk}^0| |\xi_n|^{j+k-1} \right) \sum_{j,k \geq 1} (|a_{jk}^0| |\xi_\ell|^{j+k-1})^{n-1} e^{-\frac{c_0}{4} \frac{|\xi_\ell|^2}{r_\ell}} d\xi_n \cdots d\xi_1.
\end{aligned}$$

By (7) and the fact that $j, k \geq 1$,

$$e^{-\frac{c_0}{4} \frac{|\xi_n|^2}{r_n}} |\xi_n|^{j+k-1} \leq \left(\frac{2(j+k-1)r_n}{c_0 e} \right)^{\frac{j+k-1}{2}} \leq \left(\frac{2(j+k-1)s}{c_0 e} \right)^{\frac{j+k-1}{2}} \frac{r_n^{1/2}}{s^{1/2}}$$

(where the last inequality uses $r_n \leq s$) and

$$e^{-\frac{c_0}{4} \frac{|\xi_\ell|^2}{r_\ell}} |\xi_\ell|^{(n-1)(j+k-1)} \leq \left(\frac{2(n-1)(j+k-1)s}{c_0 e} \right)^{(n-1)\frac{j+k-1}{2}}$$

Consequently,

$$\begin{aligned}
& \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |e(0, \xi_n)| |e(0, \xi_\ell)|^{n-1} d\xi_n \cdots d\xi_1 \\
& \leq A^n \frac{e^{-\frac{c_0}{4} \frac{|z|^2}{s}}}{s^{1/2}} \sum_{j,k \geq 1} (|a_{jk}^0| n s^{\frac{j+k-1}{2}} n^{\frac{j+k-1}{2}}) \frac{(s-r_1)r_1}{s} \left(\prod_{m=1}^{n-1} \frac{(r_m - r_{m+1})r_{m+1}}{r_m} \right) r_n^{1/2}
\end{aligned}$$

Proceeding as before and integrating the time derivatives using (27) yields the estimate (iii) for $\bar{W}_{\tau p, w} Z_{\tau p, z} (M_{\tau p}^{z, w})^n H_{\tau p}(s, z, w)$ with no decay in τ .

7.4. Proof of Theorem 5.1, (iii) with decay in τ . To prove the estimates with decay in τ , we estimate the space integral first. We use Lemma 7.2 on $\prod_{j=1}^{n-3} |\xi(0, \xi_j)|$ and thus we must estimate the following term.

$$\begin{aligned}
IV &:= \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |e(0, \xi_\ell)|^{n-3} |e(0, \xi_{n-2}) e(0, \xi_{n-1}) e(0, \xi_n)| d\xi_n \cdots d\xi_1 \\
& \leq \int_{\mathbb{C}^{n-2}} e^{-\frac{c_0}{2} \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-3} e^{-\frac{c_0}{2} \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-\frac{c_0}{2} \frac{|\xi_{n-2}|^2}{r_{n-2}}} |e(0, \xi_\ell)|^{n-3} |e(0, \xi_{n-2})| \\
& \quad \times \int_{\mathbb{C}^2} \left[\prod_{\alpha=n-1}^n e^{-\frac{c_0}{2} \frac{r_{\alpha-1}}{(r_{\alpha-1} - r_\alpha)r_\alpha} |\xi_\alpha - \frac{r_\alpha}{r_{\alpha-1}} \xi_{\alpha-1}|^2} \right] |e(0, \xi_{n-1})| e^{-\frac{c_0}{4} \frac{|\xi_{n-1}|^2}{r_{n-1}}} e^{-c_0 \frac{r_n}{\mu(0,1/\tau)^2}} e^{-c_0 \frac{r_{n-2}-r_n}{\mu(\xi_{n-1}, 1/\tau)^2}} \\
& \quad \times |e(0, \xi_n)| e^{-\frac{c_0}{4} \frac{|\xi_n|^2}{r_n}} e^{-c_0 \frac{r_{n-1}}{\mu(\xi_n, 1/\tau)^2}} d\xi_n \cdots d\xi_1
\end{aligned}$$

Using arguments similar to the ones used in Section 7.2 we can estimate

$$|e(0, \xi_\ell)|^{n-3} e^{-c_0 \frac{|\xi_\ell|^2}{4r_\ell}} \leq \sum_{j,k \geq 1} |a_{jk}^0|^{n-3} s^{(n-3)\frac{j+k-1}{2}} (n-3)^{(n-3)\frac{j+k-1}{2}},$$

$$|e(0, \xi_{n-2})| e^{-c_0 \frac{|\xi_{n-2}|^2}{4r_{n-2}}} \leq \sum_{j,k \geq 1} |a_{jk}^0| r_{n-2}^{\frac{j+k-1}{2}} \leq \sum_{j,k \geq 1} |a_{jk}^0| s^{(\frac{j+k-1}{2})} \frac{r_{n-2}^{1/2}}{s^{1/2}}$$

and

$$|e(0, \xi_n)| e^{-\frac{c_0}{4} \frac{|\xi_n|^2}{r_n}} e^{-c_0 \frac{r_{n-1}}{\mu(\xi_n, 1/\tau)^2}} \leq C \sum_{j \geq 1} |a_{j1}^{\xi_n}| |\xi_n|^j \frac{r_n^{\frac{j}{2}} r_{n-1}^{\frac{1}{2}}}{|\xi_n|^j r_{n-1}^{\frac{1}{2}}} \frac{\mu(\xi_n, 1/\tau)^{j+1}}{r_{n-1}^{\frac{j+1}{2}}} \leq \frac{C}{\tau} \frac{r_n^{1/4}}{r_{n-1}^{3/4}}.$$

Since $\max\{r_n, r_{n-2} - r_n\} \geq \frac{1}{2}r_{n-2}$, a similar argument shows that

$$|e(0, \xi_{n-1})| e^{-\frac{c_0}{4} \frac{|\xi_{n-1}|^2}{r_{n-1}}} e^{-c_0 \frac{r_n}{\mu(0, 1/\tau)^2}} e^{-c_0 \frac{r_{n-2} - r_n}{\mu(\xi_{n-1}, 1/\tau)^2}} \leq \frac{C}{\tau} \frac{r_{n-1}^{1/4}}{r_{n-2}^{3/4}}.$$

Thus, the space integral is estimated as follows (note that we are using the integral estimates from the earlier case with $n-2$ replacing n):

$$\begin{aligned} & \int_{\mathbb{C}^n} e^{-c_0 \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-1} e^{-c_0 \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-c_0 \frac{|\xi_n|^2}{r_n}} |e(0, \xi_\ell)|^{n-3} |e(0, \xi_{n-2})| |e(0, \xi_{n-1})| |e(0, \xi_n)| d\xi_n \cdots d\xi_1 \\ & \leq \left(\prod_{m=n-1}^n \frac{(r_{m-1} - r_m) r_m}{r_{m-1}} \right) r_n^{1/4} r_{n-1}^{-1/2} r_{n-2}^{-3/4} \\ & \times \int_{\mathbb{C}^{n-2}} e^{-\frac{c_0}{2} \frac{|z-\xi_1|^2}{s-r_1}} \left[\prod_{m=1}^{n-3} e^{-\frac{c_0}{2} \frac{|\xi_m - \xi_{m+1}|^2}{r_m - r_{m+1}}} \right] e^{-\frac{c_0}{2} \frac{|\xi_{n-2}|^2}{r_{n-2}}} |e(0, \xi_\ell)|^{n-3} |e(0, \xi_{n-2})| d\xi_{n-2} \cdots d\xi_1 \\ & \leq \frac{C^n}{\tau^2 s^{1/2}} e^{-c_0 \frac{|z|^2}{8s}} \sum_{j,k \geq 1} |a_{jk}^0|^{n-2} s^{(n-2)\frac{j+k-1}{2}} (n-2)^{(n-2)\frac{j+k-1}{2}} \frac{(s-r_1)r_1}{s} \left(\prod_{m=1}^{n-1} \frac{(r_m - r_{m+1})r_{m+1}}{r_m} \right) \frac{r_n^{1/4}}{r_{n-2}^{1/4} r_{n-1}^{1/2}}. \end{aligned}$$

We can handle the time (r) -integrals using (27) and compute

$$\begin{aligned} & \int_0^s \cdots \int_0^{r_{n-1}} (s-r_1)^{-1/2} (r_1-r_2)^{-1/2} \cdots (r_{n-1}-r_n)^{-1/2} r_{n-2}^{-1/4} r_{n-1}^{-1/2} r_n^{-3/4} dr_n \cdots dr_1 \\ & = \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \int_0^s \cdots \int_0^{r_{n-2}} (s-r_1)^{-1/2} (r_1-r_2)^{-1/2} \cdots (r_{n-2}-r_{n-1})^{-1/2} r_{n-2}^{-1/4} r_{n-1}^{-3/4} dr_{n-1} \cdots dr_1 \\ & = \frac{\pi \Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2} \int_0^s \cdots \int_0^{r_{n-3}} (s-r_1)^{-1/2} (r_1-r_2)^{-1/2} \cdots (r_{n-3}-r_{n-2})^{-1/2} r_{n-2}^{-1/2} dr_{n-2} \cdots dr_1 \\ & = \frac{\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\pi \Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2} s^{\frac{n-1}{2}-1}. \end{aligned}$$

Plugging in the space and time estimates into III finishes the τ -decay argument.

7.5. End of the proof of Theorem 5.1. The argument for (ii) follows from the arguments that we have already done. For example, if $X = \overline{Z}_{\tau p, z}$ or $Z_{\tau p, z}$, then the argument for (i) can be followed line by line. If $X = \overline{W}_{\tau p, w}$ or $W_{\tau p, w}$, then the argument to prove (iii) can be imitated line by line. Thus, all that remains is to prove the estimate for $(M_{\tau p}^{z, w})^n \frac{\partial H_{\tau p}}{\partial s}(s, z, w) = -W_{\tau p, w} \overline{W}_{\tau p, w} (M_{\tau p}^{z, w})^n H_{\tau p}(s, z, w)$. The issue is that none of tricks that we used earlier will work because the integral in r_n will not converge. Instead, we want to integrate by parts on the $Z_{\tau p, \xi}$ terms. The clean way to do this is to use the first line in (22) and integrate by parts. In the term that we have been using as our demonstration estimation, the integral (analogous to (28) above) becomes

$$\begin{aligned}
& \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} H_{\tau p}(s - r_1, z, \xi_1) \left(\prod_{j=1}^{n-1} Z_{\tau p, \xi_j} [e(0, \xi_j) H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1})] \right) \\
& \quad \times Z_{\tau p, \xi_n} [e(0, \xi_n) W_{\tau p, w} \overline{W}_{\tau p, w} H_{\tau p}(r_n, \xi_n, 0)] d\xi_n \cdots d\xi_1 dr_n \cdots dr_1 \\
(30) \quad & = (-1)^n \int_0^s \int_0^{r_1} \cdots \int_0^{r_{n-1}} \int_{\mathbb{C}^n} W_{\tau p, \xi_1} H_{\tau p}(s - r_1, z, \xi_1) \left(\prod_{j=1}^{n-1} e(0, \xi_j) W_{\tau p, \xi_{j+1}} H_{\tau p}(r_j - r_{j+1}, \xi_j, \xi_{j+1}) \right) \\
& \quad \times e(0, \xi_n) W_{\tau p, w} \overline{W}_{\tau p, w} H_{\tau p}(r_n, \xi_n, 0) d\xi_n \cdots d\xi_1 dr_n \cdots dr_1.
\end{aligned}$$

After this integration by parts, we can proceed as with (iii) above. To handle the terms that arise when the w -derivative does not get applied to $H_{\tau p}(r_n, \xi_n, 0)$, we can use a combination of integration by parts as in (30) (this will be only be needed if $X^2 = W_{\tau p, w} \overline{W}_{\tau p, w}$) and isolating the $\partial N / \partial w$ term similarly to how we handled $|e(0, \xi_{n-2})|$ in §7.4. This concludes the proof of Theorem 5.1.

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